Problem Sheet 6

1. From Theorem 6.12 we have

$$\zeta(s) = 1 + \frac{1}{s-1} - s\int_1^\infty \frac{\{u\}}{u^{1+s}}du, \tag{48}$$

valid for $\mathrm{Re}\, s > 0$.

i) Deduce that

$$\zeta(s) = s\int_1^\infty \frac{[u]}{u^{1+s}}du$$

for $\mathrm{Re}\, s > 1$.

Note the integral contains $[u]$ in place of $\{u\}$.

ii) Deduce that

$$\zeta(s) = -s\int_0^\infty \frac{\{u\}}{u^{1+s}}du,$$

for $0 < \mathrm{Re}\, s < 1$.

Note how the integral runs from 0 and not 1.

iii) Deduce from (48) that for real $\sigma > 0, \sigma \neq 1$ we have

$$\frac{1}{\sigma-1} < \zeta(\sigma) < \frac{\sigma}{\sigma-1}.$$

In particular, $\zeta(\sigma) < 0$ for $0 < \sigma < 1$.

Hint for Part iii) Use $0 \leq \{u\} < 1$.

2. Let $a_n \in \mathbb{C}$ be a sequence of coefficients and set $A(x) = \sum_{1\leq n\leq x} a_n$.

i) Use Partial Summation to prove

$$\sum_{n=1}^N \frac{a_n}{n^s} = \frac{A(N)}{N^s} + s\int_1^N A(t)\frac{dt}{t^{1+s}}, \tag{49}$$

ii) Assume that there exists a constant $C > 0$ such that $|A(x)| \leq C$ for all $x > 1$. Prove that the Dirichlet Series

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges for $\operatorname{Re} s > 0$ and satisfies

$$|F(s)| \leq C\frac{|s|}{\sigma}$$

there.

3. i) Prove, using the previous question, that the Dirichlet Series

$$F(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

converges for $\operatorname{Re} s > 0$.

ii) For the Dirichlet series $F(s)$ defined in Part i, prove that

$$F(s) = \left(1 - \frac{1}{2^{s-1}}\right)\zeta(s)$$

for $\operatorname{Re} s > 1$.

Note that we can now use part ii to define $\zeta(s)$ for $\operatorname{Re} s > 0$, $s \neq 1$, by

$$\zeta(s) = \left(1 - \frac{1}{2^{s-1}}\right)^{-1} F(s). \tag{50}$$

In this way we have a continuation of $\zeta(s)$ to the larger half plane $\operatorname{Re} s > 0$.

Hint: For Part ii consider the partial sums

$$\sum_{n=1}^{2N} \frac{(-1)^{n+1}}{n^s} \quad \text{and} \quad \sum_{n=1}^{2N} \frac{1}{n^s},$$

expressing each as sums over even and odd integers. Combine and then let $N \to \infty$.

4. Look at the proof of

$$\sum_{n \le x} \frac{1}{n} = \log n + \gamma + O\left(\frac{1}{x}\right), \tag{51}$$

to find an expression for γ, Euler's constant, which, with (48) seen in Question 1, gives a proof of

$$\lim_{s \to 1} \left(\zeta(s) - \frac{1}{s-1}\right) = \gamma.$$

5. i) Prove that

$$\lim_{s \to 1} \left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right) = \gamma.$$

Hint Writing $\zeta(s) = g(s)/(s-1)$ show that $g(1) = 1$ and, by using Question 4, $g'(1) = \gamma$.

ii) Prove that

$$\lim_{s \to 1} \left(\frac{\zeta'(s)}{\zeta(s)} + \zeta(s)\right) = 2\gamma$$

6. Show that

$$\zeta^{(\ell)}(s) = \frac{(-1)^\ell \, \ell!}{(s-1)^{\ell+1}} + (-1)^\ell \int_1^\infty \{u\} \frac{\ell \log^{\ell-1} u - s \log^\ell u}{u^{s+1}} du$$

for $\mathrm{Re}\, s > 1$.

Hint Do not attempt to differentiate (48) ℓ times, for there is then the question of how to take a derivative inside an integral. Instead use the method used in lectures when the $\ell = 1$ case was proved.

7. On Problem Sheet 2 you are asked to generalise

$$\sum_{n \le N} \frac{1}{n} = \log N + \gamma + O\left(\frac{1}{N}\right) \tag{52}$$

and prove that for all $\ell \geq 0$ there exists a constant C_ℓ such that

$$\sum_{n\leq N} \frac{\log^\ell n}{n} = \frac{1}{\ell+1} \log^{\ell+1} N + C_\ell + O\left(\frac{\log^\ell N}{N}\right),$$

for integer N. So $C_0 = \gamma$.

The Riemann zeta function has a ***Laurent Expansion*** at $s = 1$. This is a Taylor series with a finite number of negative powers allowed, and for the Riemann zeta function looks like

$$\zeta(s) = \frac{1}{s-1} + \sum_{k=0}^{\infty} c_k \left(s-1\right)^k,$$

for s close to 1, for some coefficients c_k, $k \geq 0$.

From Question 4 we have $c_0 = \gamma = C_0$. Generalise this and prove that

$$c_\ell = (-1)^\ell \frac{C_\ell}{\ell 1},$$

for $\ell \geq 1$.

Hint Differentiate the Laurent Expansion sufficiently often to get a formula for c_ℓ as a limit as $s \to 1$. Then use Question 6 along with an expression for C_ℓ found on Problem Sheet 2.

8. i) Prove that

$$5 + 8\cos\theta + 4\cos 2\theta + \cos 3\theta \geq 0, \tag{53}$$

for all θ.

ii) Deduce that

$$\zeta^5(\sigma) \left|\zeta(\sigma+it)\right|^8 \left|\zeta(\sigma+2it)\right|^4 \left|\zeta(\sigma+3it)\right| \geq 1.$$

Thus the results in Lemmas 6.19 and 6.20 are not the only ones of their type. Can you find others?

Note that (53) has a property in common with Lemma 6.19, namely the polynomials are zero when $\theta = \pi$.

9. You cannot put $s = 1$ into Theorem 6.11:

$$\sum_{1 \le n \le N} \frac{1}{n^s} = 1 + \frac{1}{s-1} + \frac{N^{1-s}}{1-s} - s \int_1^N \{u\} \frac{du}{u^{s+1}},$$

because of the $s-1$ on the denominator. Instead, what is the limit as $s \to 1$, of these two terms with $s-1$ in their denominator, i.e.

$$\lim_{s \to 1} \left(\frac{1}{s-1} + \frac{N^{1-s}}{1-s} \right)?$$

In this way give an alternative proof of

$$\sum_{1 \le n \le N} \frac{1}{n} = \log N + 1 - \int_1^N \{u\} \frac{du}{u^2}.$$

10. Prove Theorem 6.27, but only for $\sigma \ge 1$ and $t > 2$, when

$$|\zeta'(\sigma + it)| \le (\log t + 7/4)^2 .$$

Hint Estimate each term in (32) :

$$\zeta'(s) = -\sum_{n=1}^{N} \frac{\log n}{n^s} - \frac{N^{1-s} \log N}{s-1} - \frac{N^{1-s}}{(s-1)^2} - I_1(s) + sI_2(s),$$

where

$$I_1(s) = \int_N^\infty \frac{\{u\}}{u^{s+1}} du \quad \text{and} \quad I_2(s) = \int_N^\infty \frac{\{u\} \log u}{u^{s+1}} du.$$

11. Results in the lectures concern the size of the Riemann zeta function for $\operatorname{Re} s \ge 1$. In this question we go to the line $\operatorname{Re} s = 1/2$.

Prove that

$$|\zeta(1/2 + it)| \le 4t^{1/2} + 1$$

for $|t| \ge 4$.

Hint Follow the proof of Theorem 6.25, again making use of Theorem 6.24.

Aside It is expected that $\zeta(1/2 + it) \ll t^{\varepsilon}$ for sufficiently large t for all $\varepsilon > 0$, i.e. it grows smaller than any power of t we go to infinity along the line $\operatorname{Re} s = 1/2$. There is a great interest in reducing the exponent 1/2 above.

12. Assume that the Dirichlet Series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

converges at $s_0 \in \mathbb{C}$.

i. Prove that the series converges in the half plane *strictly* to the right of s_0, i.e. for all s with $\operatorname{Re} s > \operatorname{Re} s_0$.

ii. Deduce that the Riemann zeta function **diverges** for all $\operatorname{Re} s < 1$.

Note this still leaves open the question of convergence **on** $\operatorname{Re} s = 1$.

Hint For the first part show that

$$\sum_{1 \le n \le N} \frac{a_n}{n^s} = \frac{1}{N^{s-s_0}} \sum_{1 \le n \le N} \frac{a_n}{n^{s_0}} - (s - s_0) \int_1^N \sum_{1 \le n \le t} \frac{a_n}{n^{s_0}} \frac{dt}{t^{s-s_0+1}}.$$

For the deduction concerning divergence of the Riemann zeta function use proof by contradiction, the contradiction coming from the known fact that $\sum_{n=1}^{\infty} 1/n^{\eta}$ diverges for any *real* $\eta < 1$.